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theorem:

if $\lambda_1 - \lambda_2$ is not zero and a positive integer, solution basis is $X = (x_1, x_2)$

$$x_1 = |t|^{\lambda_1} p_1(t) \qquad x_2 = |t|^{\lambda_2} p_2(t) \quad , \quad 0 < |t| < \rho$$

$$p_i = 1 + \sum_{k=1} p_{i,k} t^k$$

if $\lambda_1 = \lambda_2$,

$$x_1 = |t|^{\lambda_1} p_1(t)$$

$$x_2 = |t|^{\lambda_1} p_1(t) \log|t| + |t|^{\lambda_1} p_2(t)$$

$$= x_1 \log|t| + |t|^{\lambda_1} p_2(t)$$

$$p_2(t) = \sum p_{2,k} t^k$$

Example:

$$t^2 x'' + \frac{3}{2} t x' + t x = 0 \quad , \quad t \neq 0$$

$$p_R(\lambda) = \lambda(\lambda - 1) + \frac{3}{2}\lambda = \lambda\left(\lambda + \frac{1}{2}\right) \qquad \because b_0 = 0$$

$$\therefore x(t) = |t|^{\lambda} \sum c_k t^k \quad , \quad c_0 = 1$$

$$|t|^{\lambda} \sum_{k=2} k(k-1) c_k t^k + \frac{3}{2}|t|^{\lambda} \sum_{k=1} k c_k t^k + \sum c_k t^{k+1}$$

$$+ \lambda(\lambda-1)|t|^{\lambda} \sum c_k t^k + \frac{3}{2}\lambda|t|^{\lambda} \sum c_k t^k + 2\lambda |t|^{\lambda} \sum_{k=1} k c_k t^k = 0$$

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$$p_R(\lambda)|t|^\lambda + |t|^\lambda \sum_{k=1} k(k+1)c_{k+1}t^{k+1} + |t|^\lambda \sum_{k=1} k c_k t^k \cdot \frac{3}{2}$$

$$+|t|^\lambda \sum_{k=1} c_{k-1}t^k + \lambda(\lambda-1)|t|^\lambda \sum_{k=1} c_k t^k + \frac{3}{2}\lambda|t|^\lambda \sum_{k=1} c_k t^k$$

$$+2\lambda|t|^\lambda \sum_{k=1} k c_k t^k = 0$$

$k=1$, $\frac{3}{2}c_1 + c_0 + \lambda(\lambda-1)c_1 + \frac{3}{2}\lambda c_1 + 2\lambda c_1 = 0$

$$\lambda(\lambda+\tfrac{5}{2})c_1 + \tfrac{3}{2}c_1 + c_0 = 0$$

$$\therefore (\lambda+1)(\lambda+\tfrac{3}{2})c_1 + c_0 = 0$$

$$p_R(\lambda+1)c_1 + c_0 = 0$$

$k \geq 2$, $k(k-1)c_k + \frac{3}{2}kc_k + c_{k-1} + \lambda(\lambda-1)c_k$

$$+\tfrac{3}{2}\lambda c_k + 2\lambda k c_k = 0$$

$$[\lambda(\lambda-1) + k(k-1) + 2\lambda k + \tfrac{3}{2}\lambda + \tfrac{3}{2}k]c_k + c_{k-1} = 0$$

$$[\lambda(\lambda+\tfrac{1}{2}) + k(k+\tfrac{1}{2}) + 2\lambda k]c_k + c_{k-1} = 0$$

$$\therefore [(\lambda+k)(\lambda+k+\tfrac{1}{2})]c_k + c_{k-1} = 0$$

$$p_R(\lambda+k)c_k + c_{k-1} = 0$$

$$\therefore p_R(\lambda)|t|^\lambda + |t|^\lambda \sum_{k=1}[p_R(\lambda+k)c_k + c_{k-1}]t^k = 0$$

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$$c_k = \frac{(-1)^k}{q(\lambda+1)\cdots q(\lambda+k)}, \quad k=1,2,\dots$$

$\lambda = 0$,

$$x_1(t) = 1 + \sum_{k=1} \frac{(-1)^k t^k}{q(1)\cdots q(k)}$$

$\lambda = -\frac{1}{2}$,

$$x_2(t) = \left[1 + \sum_{k=1} \frac{(-1)^k t^k}{q(\frac{1}{2})\cdots q(k-\frac{1}{2})}\right]|t|^{-\frac{1}{2}}$$

to prove that $x_1(t)$, $x_2(t)$ are independent, let's assume not,

$$a_1 x_1(t) + a_2 x_2(t) = 0$$

$$|t|^{\frac{1}{2}} a_1 x_1(t) + |t|^{\frac{1}{2}} a_2 x_2(t) = 0, \quad t \neq 0$$

let $t \to 0$, $a_2 \equiv 0$, $\therefore a_1 x_1(t) = 0$, let $t \to 0$ again,

$\therefore a_1 = 0$.

$\lambda_1 - \lambda_2 = m$, m is an integer

based on the previous theorem, the equation system can be transformed to one in which $\lambda_1 = \lambda_2$, thus, we can guess the solution basis should be

$$x_1(t) = |t|^{\lambda_1} p_1(t), \quad p_1(t) = 1 + \sum_{k=1} p_{1,k} t^k$$

from ~~①~~③ $x_2(t) = |t|^{\lambda_2}[q_1(t)\log|t| + q_2(t)]$
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remember that the Wronskian is

$$W_x(t) = W_x(t_0)\, e^{\int_{t_0}^{t} tr(A)\,ds} \qquad X' = AX$$

for the second-order equation,

$$z' = \frac{1}{t-\tau} B z \qquad B(t) = \begin{pmatrix} 0 & 1 \\ -b(t) & 1-a(\tau) \end{pmatrix}$$

∴ the Wronskian can be written as

$$W_x(t) = W_x(t_0)\, e^{-\int_{t_0}^{t} \frac{a(s)}{s} ds}$$

$$a(s) = a_0 + a_1 s + a_2 s^2 + \cdots = a_0 + s \cdot \alpha(s)$$

$$\therefore W_x(t) = W_x(t_0)\, e^{-a_0 \int_{t_0}^{t} \frac{ds}{s}}\, e^{-\int_{t_0}^{t} \alpha(s) ds} \quad \text{analytic at } t_0$$

$$= W_x(t_0)\, e^{-a_0(\log t - \log t_0)}\, e^{-\int_{t_0}^{t} \alpha(s) ds}$$

$$= W_x(t_0)\, t_0^{a_0} \cdot t^{-a_0} \cdot e^{-\int_{t_0}^{t} \alpha(s) ds}$$

$$= K t^{-a_0} r(t) \quad , \quad K = W_x(t_0) t_0^{a_0} \neq 0$$

Since $x_1(t) = t^{\lambda_1} p_1(t)$

$$x_2(t) = t^{\lambda_2} [q_1(t) \log t + q_2(t)]$$

substitute them into the differential equation,

$$W_x(t) = x_1(t)\, x_2'(t) - x_2'(t)\, x_1'(t)$$

$$= t^{\lambda_1} p_1(t) [\lambda_2 t^{\lambda_2 - 1} q_1(t) \log t + t^{\lambda_2} q_1'(t) \log t$$

$$+ t^{\lambda_2 - 1} q_1(t) + \lambda_2 t^{\lambda_2 - 1} q_2(t) + t^{\lambda_2} q_2'(t)]$$

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$$-t^{\lambda_1}[q_1(t)\log t + q_2(t)][\lambda_1 t^{\lambda_1 - 1} p_1(t) + t^{\lambda_1} p_1'(t)]$$

$$= t^{\lambda_1+\lambda_2-1}\Big[[\lambda_2 p_1 q_1 + t p_1 q_1' - \lambda_1 p_1 q_1 - t q_1 p_1']\log t$$

$$+ [p_1 q_1 + \lambda_2 p_1 q_2 + t p_1 q_2' - \lambda_1 p_1 q_2 - t p_1' q_2]\Big]$$

$$= t^{\lambda_1+\lambda_2-1}\Big[[t p_1 q_1' - (m p_1 + t p_1') q_1]\log t$$

$$+ [t p_1 q_2' - (m p_1 + t p_1') q_2 + p_1 q_1]\Big]$$

$$= t^{\lambda_1+\lambda_2-1}[r_1(t)\log t + r_2(t)]$$

$$r_1(t) = t p_1(t) q_1'(t) - (m p_1(t) + t p_1'(t)) q_1(t)$$

$$r_2(t) = t p_1(t) q_2'(t) - (m p_1(t) + t p_1'(t)) q_2(t) + p_1(t) q_1(t)$$

$$\lambda_1 - \lambda_2 = m$$

since $P_R(\lambda) = \lambda(\lambda-1) + a_0\lambda + b_0 = (\lambda-\lambda_1)(\lambda-\lambda_2)$

$$= \lambda^2 - (\lambda_1+\lambda_2)\lambda + \lambda_1\lambda_2$$

$$\therefore -a_0 = \lambda_1 + \lambda_2 - 1$$

hence, $r_1(t)\log t + r_2(t) = K \cancel{t^{a_0}} r(t)$

as $t \to 0^+$,

$r_2(t)$ and $t^{-a_0} r(t)$ converge to some finite limits.

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if $r_1(t)$ also converges to a finite limit, contradiction will occur because $\log t \to \infty$.

Specifically, if c_k in $r_1(t) = \sum c_k t^k$ is the smallest coeff. that is non-zero, differentiating k times,

$$k! u_1(t) \log t + \frac{k}{t} r_1^{(k-1)}(t) + \dots + \frac{(-1)^{k-1}(k-1)!}{t^k} r_1(t)$$

$$+ r_2^{(k)}(t) = k' r^k(t)$$

where $r_1(t) = t^k u_1(t)$, $u_1(t)$ is analytic at $t=0$, only the first term goes to infinity.

$\therefore \; r_1(t) = 0 = t p_1 q_1' - t p_1' q_1 - m p_1 q_1$

note that $\left(\frac{q_1}{p_1}\right)' = \frac{q_1'}{p_1} - \frac{q_1 p_1'}{p_1^2}$

$$\left(\frac{1}{t^m}\right)' = -\frac{m}{t^{m+1}}$$

$$\therefore \; 0 = \frac{1}{p_1^2 t^{m+1}} \left(t p_1 q_1' - t p_1' q_1 - m p_1 q_1 \right)$$

$$= \left(\frac{q_1}{t^m p_1} \right)'$$

$$\therefore \; q_1 = c t^m p_1$$

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$$\therefore x_2(t) = |t|^{\lambda_2}(|t|^{m} c p_1(t) \log|t| + q_2(t))$$

$$= |t|^{\lambda_1} c p_1(t) \log|t| + q_2(t) \cdot |t|^{\lambda_2}$$

$$\boxed{x_2(t) = c x_1(t) \log|t| + |t|^{\lambda_2} q_2(t)}$$ if $\lambda_1 - \lambda_2$ is an integer

The Bessel equation

$$t^2 x'' + t x' + (t^2 - \alpha^2)x = 0$$

$$P_R(\lambda) = \lambda(\lambda - 1) + \lambda - \alpha^2 = \lambda^2 - \alpha^2$$

$$\therefore \lambda_1 = \alpha, \quad \lambda_2 = -\alpha$$

assuming that $\mathrm{Re}(\alpha) \geq 0$, $\mathrm{Re}(\lambda_1) \geq \mathrm{Re}(\lambda_2)$

note that $a(t) = 1 = a_0$

$b(t) = -\alpha^2 + t^2, \quad b_0 = -\alpha^2$

$$\therefore x_1(t) = |t|^{\alpha} \sum c_k t^k, \quad c_0 \neq 0$$

$$x_1'(t) = \alpha |t|^{\alpha - 1} \sum c_k t^k + |t|^{\alpha} \sum_{k=1} k c_k t^{k-1}$$

$$x_1''(t) = \alpha(\alpha - 1)|t|^{\alpha - 2} \sum c_k t^k + 2\alpha |t|^{\alpha - 1} \sum_{k=1} k c_k t^{k-1}$$

$$+ |t|^{\alpha} \sum_{k=2} k(k-1) c_k t^{k-2}$$

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$\therefore L x_1(t)$

$$= \alpha(\alpha-1)|t|^\alpha \sum c_k t^k + 2\alpha|t|^\alpha \sum_{k=1} k c_k t^k + |t|^\alpha \sum_{k=2} k(k-1) c_k t^k$$
$$+ \alpha|t|^\alpha \sum c_k t^k + |t|^\alpha \sum_{k=1} k c_k t^k + |t|^\alpha \sum c_k t^{k+2}$$
$$- \alpha^2 |t|^\alpha \sum c_k t^k$$

case 1 : $\alpha = 0$

$$\sum_{k=2} k(k-1) c_k t^k + \sum_{k=1} k c_k t^k + \sum c_k t^{k+2} = 0$$

divide by t,

$$0 = \sum_{k=2} k(k-1) c_k t^{k-1} + c_1 + \sum_{k=2} k c_k t^{k-1} + \sum c_k t^{k+1}$$

$$= c_1 + \sum_{k=1} [k(k+1) c_{k+1} + (k+1) c_{k+1} + c_{k-1}] t^k$$

$$\therefore c_1 = 0, \quad (k+1)^2 c_{k+1} + c_{k-1} = 0$$

$$\therefore c_3 = c_5 = c_7 = \cdots = 0, \quad c_2 = \frac{-c_0}{2^2} \equiv \frac{-1}{2^2}$$

$$c_4 = -\frac{c_2}{4^2}, \quad c_6 = -\frac{c_4}{6^2}$$
$$= \frac{1}{4^2 \cdot 2^2} \qquad = \frac{-1}{2^2 4^2 6^2}$$

$$\therefore c_{2m} = \frac{(-1)^m}{2^2 4^2 \cdots (2m)^2} = \frac{(-1)^m}{2^{2m} (m!)^2}$$

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Bessel function of the first kind of order zero :

$$J_0(t) = \sum \frac{(-1)^m}{2^{2m}(m!)^2} t^{2m}$$

to find the second solution, we know that

$$x_2(t) = J_0(t)\log(t) + p_2(t) \quad , \quad p_2(t) = \sum d_k t^k$$

$$x_2'(t) = J_0'(t)\log(t) + \frac{1}{t} J_0(t) + \sum_{k=1} d_k \cdot k t^{k-1}$$

$$x_2''(t) = J_0''(t)\log(t) + \frac{2}{t} J_0'(t) - \frac{1}{t^2} J_0(t) + \sum_{k=2} k(k-1) d_k t^{k-2}$$

$\therefore$ since $\alpha = 0$, $L x_2(t) = t x_2''(t) + x_2'(t) + t x_2(t) \equiv 0$

thus, $L(x_2(t)) = \log(t) L J_0(t) + \sum_{k=2} k(k-1) d_k t^{k-1} + \sum_{k=1} d_k \cdot k t^{k-1}$

$$+ \sum d_k t^{k+1} + 2J_0'(t)$$

$$= 0 + \sum_{k=1} k(k+1) d_{k+1} t^k + d_1 + \sum_{k=2} k d_k t^{k-1}$$

$$+ \sum_{k=1} d_{k-1} t^k + 2J_0'(t) \equiv 0$$

$$\therefore \sum_{k=1} [k(k+1) d_{k+1} \; + (k+1) d_{k+1} + d_{k-1}] t^k + d_1 = -2J_0'(t)$$

$$= -2 \sum_{k=1} \frac{(-1)^k}{2^{2k}(k!)^2} 2k t^{2k-1}$$

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thm implies that

$d_1 = 0$, $(k+1)^2 d_{k+1} + d_{k-1} = 0$ if $k = 2m$

$\therefore d_1 = d_3 = d_5 = \cdots = 0$

if $k = 2m-1$, $m = 1, 2, \cdots$

$$(2m)^2 d_{2m} + d_{2m-2} = -2\frac{(-1)^m}{2^{2m}(m!)^2}\cdot 2m = \frac{(-1)^{m+1}}{2^{2m-2}(m!)^2} m$$

if we set $d_0 = 0$,

$$d_2 = \frac{1}{2^2}\cdot 1 = \frac{1}{2^2}$$

$$d_4 = \frac{1}{4^2}\left[\frac{-1}{2^2 2^2} 2 - \frac{1}{2^2}\right] = \frac{-1}{2^2 4^2}\left(1+\frac{1}{2}\right)$$

$$d_6 = \frac{1}{6^2}\left[\frac{1}{2^4 36} 3 + \frac{1}{2^2 4^2}\left(1+\frac{1}{2}\right)\right]$$

$$= \frac{1}{2^2 4^2 6^2}\left[1+\frac{1}{2}+\frac{1}{3}\right)$$

$\vdots$

$$d_{2m} = \frac{(-1)^{m-1}}{2^2\cdots(2m)^2}\left(1+\frac{1}{2}+\cdots+\frac{1}{m}\right)$$

$$= \frac{(-1)^{m-1}}{2^{2m}(m!)^2}\left(1+\frac{1}{2}+\cdots+\frac{1}{m}\right)$$

Bessel function of the second kind of order zero:

$$K_0(t) = J_0(t)\log t + \sum_{m=1} \frac{(-1)^{m-1}}{2^{2m}(m!)^2}\left(1+\frac{1}{2}+\cdots+\frac{1}{m}\right)t^{2m}$$

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Case 2 : $\alpha \neq 0$, $\alpha \neq$ integer

$$\lambda_1 = \alpha, \quad \lambda_2 = -\alpha, \quad Re(\lambda_1) \geq Re(\lambda_2)$$

$$x_1(t) = |t|^{\alpha} \sum c_k t^k$$

$$L x_1(t) = 2\alpha |t|^{\alpha} \sum_{k=1} k c_k t^k + |t|^{\alpha} \sum_{k=2} k(k-1) c_k t^k$$

$$+ |t|^{\alpha} \sum_{k=1} k c_k t^k + |t|^{\alpha} \sum c_k t^{k+2}$$

$$= 2\alpha c_1 |t|^{\alpha+1} + 2\alpha |t|^{\alpha} \sum_{k=2} k c_k t^k$$

$$+ |t|^{\alpha} \sum_{k=2} k(k-1) c_k t^k + c_1 |t|^{\alpha+1} + |t|^{\alpha} \sum_{k=2} c_k t^k \cdot k$$

$$+ |t|^{\alpha} \sum_{k=2} c_{k-2} t^k$$

$$= (2\alpha+1) c_1 |t|^{\alpha+1} + |t|^{\alpha} \sum_{k=1} \left[(2\alpha k + k^2) c_k + c_{k-2} \right] t^k$$

$$= \left[(\alpha+1)^2 - \alpha^2 \right] c_1 |t|^{\alpha+1} + |t|^{\alpha} \sum_{k=1} \left[\left[(\alpha+k)^2 - \alpha^2 \right] c_k + c_{k-2} \right] t^k$$

$$= 0$$

assume $t > 0$,

$$c_1 = 0$$

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$$c_k = \frac{-c_{k-2}}{k(2\alpha+k)}, \quad \therefore \quad c_1 = c_3 = c_5 = \cdots = 0$$

$$c_2 = \frac{-c_0}{2(2\alpha+2)} = \frac{-c_0}{2^2(\alpha+1)}$$

$$c_4 = \frac{-c_2}{4(2\alpha+4)} = \frac{c_0}{2^4 2!(\alpha+1)(\alpha+2)}$$

$$c_6 = \frac{-c_4}{6(2\alpha+6)} = \frac{-c_0}{2^6 3!(\alpha+1)(\alpha+2)(\alpha+3)}$$

$$\vdots$$

$$c_{2m} = \frac{(-1)^m c_0}{2^{2m} m!(\alpha+1)(\alpha+2)\cdots(\alpha+m)}$$

$$\therefore x_1(t) = c_0 t^\alpha + c_0 t^\alpha \sum_{m=1} \frac{(-1)^m}{m!(\alpha+1)(\alpha+2)\cdots(\alpha+m)}\left(\frac{t}{2}\right)^{2m}$$

Gamma function

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

$$\Gamma(1) = \int_0^\infty e^{-t} dt = \left[-e^{-t}\right]_0^\infty = 1$$

$$\Gamma(z+1) = \lim_{T \to 0} \int_0^T e^{-t} t^z dt$$

$$= \lim_{T \to 0} \left\{ \left[-e^{-t} t^z\right]_0^T + \int_0^T e^{-t} \cdot z t^{z-1} dt \right\}$$

$$= \lim_{T \to 0} \left[z \int_0^\infty e^{-t} t^{z-1} dt \right]$$

$$= z\Gamma(z)$$

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if z is a positive integer,

$$\Gamma(z) = \underset{(z-1)}{\wedge}\Gamma(z-1) = \cdots = (z-1)!$$

if z is a negative integer, or zero,

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt = \infty,$$

thus define $\frac{1}{\Gamma(z)} = 0$

let $-N < \mathrm{Re}(z) < -N+1$, $\mathrm{Re}(z)+N > 0$

$$\Gamma(z) = \frac{\Gamma(z+N)}{z(z+1)\cdots(z+N-1)}$$

since $x_1(t) = c_0 t^\alpha + c_0 t^\alpha \sum_{m=1} \frac{(-1)^m}{m!(\alpha+1)\cdots(\alpha+m)} \left(\frac{t}{2}\right)^{2m}$

if we set $c_0 = \frac{1}{2^\alpha \Gamma(\alpha+1)}$

$$x_1(t) = \frac{t^\alpha}{2^\alpha \Gamma(\alpha+1)} + \frac{t^\alpha}{2^\alpha} \sum_{m=1} \frac{(-1)^m}{m!\,\Gamma(\alpha+1)\cdot(\alpha+1)\cdots(\alpha+m)} \left(\frac{t}{2}\right)^{2m}$$

$$= \left(\frac{t}{2}\right)^\alpha \frac{1}{\Gamma(\alpha+1)} + \left(\frac{t}{2}\right)^\alpha \sum_{m=1} \frac{(-1)^m}{m!\,\Gamma(\alpha+m+1)} \left(\frac{t}{2}\right)^{2m}$$

Bessel funtion of the first kind of order α:

$$\boxed{J_\alpha(t) = \left(\frac{t}{2}\right)^\alpha \sum \frac{(-1)^m}{m!\,\Gamma(\alpha+m+1)} \left(\frac{t}{2}\right)^{2m}}$$

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$$\boxed{J_{-\alpha}(t) = \left(\frac{t}{2}\right)^{-\alpha} \sum \frac{(-1)^m}{m!\,\Gamma(-\alpha+m+1)} \left(\frac{t}{2}\right)^{2m}}$$

case 3 : $\alpha \neq 0$, $\alpha = n$ (integer)

if $\alpha = n$, J_α and $J_{-\alpha}$ are not linearly independent.

because

$$J_{-n}(t) = \left(\frac{t}{2}\right)^{-n} \sum \frac{(-1)^m}{m!\,\Gamma(m-n+1)} \left(\frac{t}{2}\right)^{2m}$$

$m-n+1 > 0 \Rightarrow m \geq n-1$, $\therefore$ m starts from n,

all terms $m < n$ are zero.

thus,

$$J_{-n}(t) = \left(\frac{t}{2}\right)^{-n} \sum_{m=n} \frac{(-1)^m}{m!\,\Gamma(m-n+1)} \left(\frac{t}{2}\right)^{2m}$$

$$= \left(\frac{t}{2}\right)^{-n} \sum_{m=0} \frac{(-1)^{m+n}}{(m+n)!\ \Gamma(m+1)} \left(\frac{t}{2}\right)^{2m+2n}$$

$$= \left(\frac{t}{2}\right)^{n} \sum \frac{(-1)^n(-1)^m}{m!\ \Gamma(m+n+1)} \left(\frac{t}{2}\right)^{2m}$$

$$= (-1)^n J_n \quad //$$

lets start from the beginning,

$x_2(t) = c J_n(t) \log(t) + t^{-n} \sum d_k t^k$

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$$x_2'(t) = c J_n'(t)\log(t) + c\frac{1}{t}J_n(t) + (k-n)t^{-n}\sum d_k t^{k-1}$$

$$x_2''(t) = c J_n''(t)\log(t) + 2c\frac{1}{t}J_n'(t) - \frac{c}{t^2}J_n(t)$$

$$+ (k-n)(k-n-1)t^{-n}\sum d_k t^{k-2}$$

substitute these into the Bessel equation

$$L(x_2(t))$$

$$= 2ct J_n'(t) + (k-n)(k-n-1)t^{-n}\sum d_k t^k + (k-n)t^{-n}\sum d_k t^k$$

$$+ (t^2 - n^2)t^{-n}\sum d_k t^k$$

$$= 0$$

$$\therefore \sum_{k=2}^{\infty}\left[(k-n)(k-n-1)d_k + (k-n)d_k - n^2 d_k + d_{k-2}\right]t^{k-n}$$

$$+ t^{-n}\left[d_0(-n)(-n-1) + d_0(-n) - d_0 n^2\right] + t^{-n+1}\left[d_1(1-n)(-n) - n^2\right.$$

$$\left. + d_1(1-n)\right]$$

$$= -2ct\left(\frac{t}{2}\right)^n \sum \frac{(-1)^k}{k!\,\Gamma(n+k+1)}\frac{(2k+n)}{2}\left(\frac{t}{2}\right)^{2k-1}$$

~~$= -c(2m+n)\left(\frac{t}{2}\right)^n \sum$~~

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$$= -2C \text{[illegible]} \left(\frac{t}{2}\right)^n \sum \frac{(-1)^k}{k!\,\Gamma(n+k+1)} \left(\frac{t}{2}\right)^{2k} \cdot (2k+n)$$

for $k < 2n$,

$$\begin{cases} [(k-n)(k-n-1) + (k-n) - n^2]\, d_k + d_{k-2} = 0 \\ [(1-n^2) - n^2]\, d_1 = [1-2n]\, d_1 = 0 \implies d_1 = 0 \\ d_0 \text{ is undetermined, because } (-n)(-n-1) - n - n^2 = 0 \end{cases}$$

$\therefore\ d_1 = d_3 = \cdots = d_{n-a} = 0$, $a = 1$ if n is even, $a = 2$ if n is odd.

thus, let $k = 2m$,

$$[(2m-n)(2m-n-1) + (2m-n) - n^2]\, d_{2m} + d_{2m-2} = 0$$

$$[(2m-n)^2 - n^2]\, d_{2m} = -d_{2m-2}$$

$$\therefore d_{2m} = \frac{-d_{2n-2}}{2m \cdot (2m-2n)} = \frac{d_{2m-2}}{2^2 m(n-m)}$$

$$d_2 = \frac{d_0}{2^2(n-1)}, \quad d_4 = \frac{d_2}{2^2 \cdot 2(n-2)} = \frac{d_0}{2^4 \cdot 2!(n-1)(n-2)}$$

$$\vdots$$

$$d_{2m} = \frac{d_0}{2^{2m} m!(n-1)\cdots(n-m)}$$

~~for $k = 2n$~~

~~$d_{2n} = 2C\left(\frac{t}{2}\right)^n \frac{1}{\Gamma(n+1)} \cdot n$~~

for $k = 2n$,

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$$[(n)(n-1)+n-n^2]d_{2n}+d_{2n-2} = -2c\cdot\frac{1}{2^n}\frac{1}{\Gamma(n+1)}\cdot n$$

$$\therefore \frac{-c}{2^{n-1}(n-1)!} = d_{2n-2} = \frac{d_0}{2^{2n-2}[(n-1)!]^2}$$

$$\therefore c = \frac{-d_0}{2^{n-1}(n-1)!}$$

we can arbitarily set $c=1$, thus

$$d_0 = -2^{n-1}(n-1)!\ ,\quad \therefore \boxed{d_{2m} = \frac{-2^{n-1}(n-1)!}{2^{2m}\,m!\,(n-1)\cdots(n-m)} = \frac{-(n-m-1)!}{2^{2m-n+1}\,m!}\ ,\ m<n}$$

<u>for $k>2n$, $k=2m+2n$,</u>

$$[(k-n)(k-n-1)+(k-n)-n^2]d_k+d_{k-2}$$
$$= -2\cdot\left(\frac{1}{2}\right)^n\cdot\frac{(-1)^m}{m!\,\Gamma(n+m+1)}\left(\frac{1}{2}\right)^{2m}(2m+n)$$

$$\therefore -\frac{1}{2^{n-1}}\frac{(-1)^m}{m!\,\Gamma(n+m+1)}\frac{2m+n}{2^{2m}}$$
$$= [(2m+n)(2m+n-1)+2m+n\underset{-n^2}{\wedge}]d_{2n+2m}+d_{2n+2m-2}$$
$$= \underset{(2m+2n)\cdot(2m)}{\cancel{\quad\quad}}\,d_{2n+2m}+d_{2n+2m-2}$$

<u>$m=0$ case:</u>

$$0\cdot d_{2n}+d_{2n-2} = -\frac{1}{2^{n-1}}\frac{1}{n!}\cdot n = -\frac{1}{2^{n-1}}\frac{1}{(n-1)!}$$

$$0\cdot d_{2n} = -\frac{1}{2^{n-1}}\frac{1}{(n-1)!}+\frac{1}{2^{n-1}(n-1)!} = 0$$!! of course.

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$m=1$, $m=2$

$$2^{4} d_{2n+2} (n+1)^{4} + d_{2n} = \frac{1}{2^{n-1}} \frac{1}{(n+1)!} \frac{2+n}{2^{2}}$$

$$2!\cdot 2^{4} d_{2n+4} (n+2)^{4} + d_{2n+2} = \frac{1}{2^{n-1}} \frac{-1}{2!(n+2)!} \frac{n+4}{2^{4}}$$

$$\therefore d_{2n+2} = \frac{1}{2^{4}}\left[\frac{1}{2^{n+1}} \frac{1}{(n+1)!}\left(1+\frac{1}{n+1}\right) - \frac{d_{2n}}{n+1}\right]$$

~~$d_{2n+4} = \frac{1}{2!\,2^{4}}\left[\frac{1}{2^{n+2}} \frac{-1}{2!(n+2)!}\left(1+\frac{1}{n+2}\right.\right.$~~

assume that $d_{2n} = \frac{-1}{2^{n+1}} \frac{1}{n!}\left(1+\cdots+\frac{1}{n}\right)$

then, $d_{2n+2} = \frac{1}{2^{2}}\left[\frac{1}{2^{n+1}} \frac{1}{(n+1)!}\left(1+\frac{1}{n+1}\right) + \frac{1}{2^{n+1}} \frac{1}{(n+1)!}\left(1+\cdots+\frac{1}{n}\right)\right]$

$$= \frac{1}{2^{2}} \frac{1}{(n+1)!} \frac{1}{2^{n+1}}\left[1+\frac{1}{n+1}+1+\cdots+\frac{1}{n}\right]$$

$$= \frac{1}{2^{n+1}} \frac{1}{(n+1)!} \frac{1}{2^{2}}\left(1+1+\cdots+\frac{1}{n+1}\right)$$

$$= \frac{1}{2^{n+1}} \frac{1}{(n+1)!} \frac{1}{2^{2}}\left(4(1)+4(n+1)\right)$$

$$d_{2n+4} = \frac{1}{2^{2}}\left[\frac{-1}{2^{n+3}} \frac{1}{2!(n+2)!}\left(\frac{n+4}{n+2}\right)\cdot\frac{1}{2} - \frac{1}{2^{n+1}} \frac{1}{(n+1)!} \frac{1}{2^{2}}\left(1+1+\cdots+\cancel{\frac{1}{n+1}}\frac{1}{n+2}\frac{1}{2}\right)\right]$$

$$= \frac{1}{2^{2}}\left[\frac{-1}{2^{n+3}} \frac{1}{2!(n+2)!}\left(\frac{1}{2}+\frac{1}{n+2}\right) - \frac{1}{2^{n+3}} \frac{1}{2!(n+2)!}\left(1+1+\cdots+\frac{1}{n+1}\right)\right]$$

$$= \frac{-1}{2^{n+3}} \frac{1}{2!(n+2)!} \frac{1}{2^{2}}\left(4(2)+4(n+2)\right)$$

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Since when $m=3$,

$$3 \cdot 2^2 d_{2n+6}(n+3) + d_{2n+4} = \frac{1}{2^{n-1}} \frac{1}{3!(n+3)!} \frac{n+6}{2^6}$$

$$d_{2n+6} = \frac{1}{2^2}\left[\frac{1}{2^{n+5}} \frac{1}{3!(n+3)!} \left(\frac{n+6}{n+3}\right) \cdot \frac{1}{3} + \frac{1}{2^{n+5}} \frac{1}{2!(n+2)!} \left(1+\frac{1}{2}+1+\dots+\frac{1}{n+2}\right) \cdot \frac{1}{n+3} \cdot \frac{1}{3} \right]$$

$$= \frac{1}{2^2} \frac{1}{2^{n+5}} \frac{1}{3!(n+3)!} \left[\frac{1}{3} + \frac{1}{n+3} + 1 + \frac{1}{2} + 1 + \dots + \frac{1}{n+2} \right]$$

$$= \frac{1}{2^{n+5}} \frac{1}{3!(n+3)!} \frac{1}{2^2} \left[\psi(3) + \psi(n+3) \right)$$

thus, there is a pattern of

$$d_{2n+2m} = \frac{(-1)^{m+1}}{2^{n+2m+1}} \frac{1}{m!(n+m)!} \left(\psi(m) + \psi(n+m) \right)$$

$$\therefore x_2(t) = J_n(t)\log(t) \cancel{t} - t^{-n} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{2^{2k-n+1}\, k!} t^{2k}$$

$$+ t^n \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2^{n+2k+1}} \frac{1}{k!(n+k)!} \left(\psi(k) + \psi(n+k) \right) t^{2k}$$

$$= J_n(t)\log(t) - \frac{1}{2}\left(\frac{t}{2}\right)^{-n} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{t}{2}\right)^{2k}$$

$$- \frac{1}{2}\left(\frac{t}{2}\right)^{n} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left[\psi(k) + \psi(n+k) \right] \left(\frac{t}{2}\right)^{2k}$$

Bessel function of the second kind of order α:

$$Y_\alpha(t) = J_\alpha(t)\log(t) - \frac{1}{2}\left(\frac{t}{2}\right)^{-\alpha}\sum_{k=0}^{\alpha-1}\frac{(\alpha-k-1)!}{k!}\left(\frac{t}{2}\right)^{2k}$$

$$- \frac{1}{2}\left(\frac{t}{2}\right)^{\alpha}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(\alpha+k)!}\left[\psi(k)+\psi(\alpha+k)\right]\left(\frac{t}{2}\right)^{2k}$$

Neumann function:

$$Y_\alpha(t) = \frac{\cos\alpha\pi \cdot J_\alpha(t) - J_{-\alpha}(t)}{\sin\alpha\pi}$$

if $\alpha = n$ (integer) $Y_\alpha(t) = 0/0$ undefined.

recall that $J_\alpha(t) = \sum \frac{(-1)^k}{k!\,\Gamma(k+\alpha+1)}\left(\frac{t}{2}\right)^{2k+\alpha}$,

and $\psi(t) = \frac{\Gamma'(t)}{\Gamma(t)}$

by l'Hospital's rule,

$$Y_n(t) = \lim_{\alpha\to n}\frac{\cos\alpha\pi\, J_\alpha(t) - J_{-\alpha}(t)}{\sin\alpha\pi}$$

$$= \lim_{\alpha\to n}\frac{-\pi\sin\alpha\pi\, J_\alpha(t) + \cos\alpha\pi\, J'_\alpha(t) - J'_{-\alpha}(t)}{\pi\cos\alpha\pi}$$

$$= \frac{1}{\pi}\left[J'_\alpha(t) - (-1)^n J'_{-\alpha}(t)\right]\Big|_{\alpha=n}$$

where $J'_\alpha(t) = \frac{d}{d\alpha}J_\alpha(t)$

note that $y = x^a$

$$\log y = a \log x$$

$$\frac{1}{y}\frac{dy}{da} = \log x$$

$$\frac{dy}{da} = x^a \log x$$

log is the natural logarithm

$$\therefore \frac{d}{d\alpha} J_\alpha(x) = \frac{d}{d\alpha}\left[\sum \frac{(-1)^k}{k!\,\Gamma(k+\alpha+1)}\left(\frac{x}{2}\right)^{2k+\alpha}\right]$$

$$= \sum \frac{(-1)^k}{k!\,\Gamma(k+\alpha+1)}\left(\frac{x}{2}\right)^{2k+\alpha} \log\left(\frac{x}{2}\right)$$

$$- \sum \frac{(-1)^k \Gamma'(k+\alpha+1)}{k!\,\Gamma^2(k+\alpha+1)}\left(\frac{x}{2}\right)^{2k+\alpha}$$

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spherical Bessel function

$$J_{-\frac{1}{2}}(t) = \sqrt{\frac{2}{t}} \sum \frac{(-1)^m}{m!\,\Gamma(m+1-\frac{1}{2})} \left(\frac{t}{2}\right)^{2m}$$

$$= \sqrt{\frac{2}{t}} \sum \frac{(-1)^m}{(2m)!\,\sqrt{\pi}} t^{2m}$$

$$= \frac{\sqrt{2}}{\sqrt{\pi t}} \sum \frac{(-1)^m}{(2m)!} t^{2m}$$

$$= \frac{\sqrt{2}}{\sqrt{\pi t}} \cos t$$

$$J_{\frac{1}{2}}(t) = \sqrt{\frac{t}{2}} \sum \frac{(-1)^m}{m!\,\Gamma(m+1+\frac{1}{2})} \left(\frac{t}{2}\right)^{2m}$$

$$= \sqrt{\frac{t}{2}} \sum \frac{(-1)^m}{(2m+1)!\,\frac{\sqrt{\pi}}{2}} t^{2m}$$

$$= \sqrt{\frac{2}{\pi t}} \sum \frac{(-1)^m}{(2m+1)!} t^{2m+1}$$

$$= \sqrt{\frac{2}{\pi t}} \sin t$$

$$2^{2m}\Gamma(m+1-\tfrac{1}{2}) = (2m-1)(2m-3)\cdots 3\cdot 1\cdot\sqrt{\pi}\cdot 2^{m}$$

$$2^{2m-2}\Gamma(m-\tfrac{1}{2}) = (2m-3)(2m-5)\cdots 3\cdot 1\cdot\sqrt{\pi}\cdot 2^{m-1}$$

~~$\frac{\Gamma(m+1-\frac{1}{2})}{\Gamma(m-\frac{1}{2})} = (2m-1) = \frac{(2m+1)2m(2m-1)}{2m}$~~

~~$\frac{\Gamma(m+1-\frac{1}{2})}{\Gamma(m-\frac{1}{2})} = 2m-1 = \frac{(2m-1)2(m-1)}{2(m-1)} = \frac{1}{2}\,(2m-1)(2m-1)(2m-3)\cdots 1$~~

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$$\frac{2^{2m}\Gamma(m+1-\frac{1}{2})}{2^{2m-2}\Gamma(m-\frac{1}{2})} = 2\cdot(2m-1) = \frac{(2m-1)(2m-2)}{m-1} = \frac{(2m-1)(2m-2)\cdots m}{(2m-3)(2m-4)\cdots m\cdot(m-1)}$$

$$\therefore 2^{2m}\Gamma(m+1-\frac{1}{2}) = \frac{(2m-1)!}{(m-1)!}\cdot\underbrace{\sqrt{\pi}\cdot 2}$$

determined by putting $m=1$

similarly.

$$2^{2m}\Gamma(m+1+\frac{1}{2}) = (2m+1)(2m-1)\cdots3\cdot1\cdot\sqrt{\pi}\cdot2^{m-1}$$

$$2^{2m-2}\Gamma(m+\frac{1}{2}) = (2m-1)(2m-3)\cdots3\cdot1\cdot\sqrt{\pi}\cdot2^{m-2}$$

$$\frac{2^{2m}\Gamma(m+1+\frac{1}{2})}{2^{2m-2}\Gamma(m+\frac{1}{2})} = 2(2m+1) = \frac{(2m+1)2m}{m} = \frac{(2m+1)(2m)\cdots(m+1)}{(2m-1)(2m-3)\cdots(m+1)m}$$

$$\therefore 2^{2m}\Gamma(m+1+\frac{1}{2}) = \frac{(2m+1)!}{m!}\sqrt{\pi}/2$$

$$t^2x'' + 2tx' + (t^2-n(n+1))X = 0$$

let $x = \frac{X}{\sqrt{t}}$

$$t^2\frac{d^2}{dt^2}\left(\frac{X}{\sqrt{t}}\right) + 2t\frac{d}{dt}\left(\frac{X}{\sqrt{t}}\right) + (t^2-n(n+1))\frac{X}{\sqrt{t}} = 0$$

$$\left(\frac{3}{4}\frac{X}{t^{1/2}} - 2\cdot\frac{1}{2}\sqrt{t}X' + t^{3/2}x''\right)$$

$$+\left(-\frac{X}{\sqrt{t}} + 2\sqrt{t}x'\right) + (t^2-n(n+1))\frac{X}{\sqrt{t}} = 0$$

$$t^2X'' + tx' + (t^2-(n+\frac{1}{2})^2)X = 0$$

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∴ by transformation $x \to \frac{x}{\sqrt{t}} = y$,

spherical Bessel → Bessel equation

if n = 0, $x = J_{\frac{1}{2}}$, $x = J_{-\frac{1}{2}}$

$= \sqrt{\frac{2}{\pi t}} \sin t$ $\quad = \sqrt{\frac{2}{\pi t}} \cos t$

$$y = \sqrt{\frac{2}{\pi}} \frac{\sin t}{t}, \quad y = \sqrt{\frac{2}{\pi}} \frac{\cos t}{t}$$

another derivation of general solutions to spherical Bessel equation when n is integer

a clever iteration method,

n = 0,

$$t^2 x'' + 2tx' + t^2 x = 0$$

by inspection, $x = \frac{\sin t}{t}$

$\therefore x' = \frac{\cos t}{t} - \frac{\sin t}{t^2}$, $x'' = -\frac{\sin t}{t} - \frac{\cos t}{t^2} - \frac{\cos t}{t^2} + \frac{2\sin t}{t^3}$

< or $(t^2 x')' + t^2 x = 0$ >

take derivative,

~~$(t^2 x'')' + (2tx')' + (t^2 x)' = 0$~~

~~$t^2 x''' + 2tx'' + 2tx'' + 2x' + 2tx + t^2 x' = 0$~~

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$$\frac{d}{dt}\left\{\frac{1}{t^2}\left[t^2x''+2tx'+t^2x\right]\right\}=0 \quad , t\neq 0$$

$$x'''+2\frac{x''}{t}-\frac{2x'}{t^2}+x'=0$$

$$t^2(x')''+2t(x')'+(t^2-2)x'=0$$

$$\therefore \quad x_0=\frac{\sin t}{t}, \quad \frac{\cos t}{t}$$

$$x_1=\frac{d}{dt}\left(\frac{\sin t}{t}\right), \quad \frac{d}{dt}\left(\frac{\cos t}{t}\right)$$

generally,

$$t^2x''+2tx'+[t^2-n(n+1)]x=0$$

taking derivative,

let $z=t^{-n}x$

$$x''=t^nz''+2nt^{n-1}z'+n(n-1)t^{n-2}z$$

$$x'=t^nz'+nt^{n-1}z$$

$$\therefore \quad t^{2+n}z''+2nt^{n+1}z'+n(n-1)t^nz$$
$$+2t^{n+1}z'+2nt^nz+t^n[t^2-n(n+1)]z=0$$

$$z''+\frac{2(n+1)}{t}z'+z=0$$

$$\therefore \quad z'''+\frac{2(n+1)}{t}z''-\frac{2(n+1)}{t^2}z'+z'=0$$

note that $\left(\frac{1}{t}z'\right)''=\frac{z'''}{t}+\frac{-2}{t^2}z''+\frac{2}{t^2}z'$

$$\left(\frac{1}{t}z'\right)'=\frac{z''}{t}-\frac{1}{t^2}z'$$